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LR tests for two hypotheses in profile analysis of growth curve data

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Abstract. This paper is concerned with profile analysis of k p-dimensional normal populations Π_i : $N_p(\boldsymbol{\mu}_i, \Sigma)$, $i = 1, \ldots, k$, when the mean vectors are expressed as $\boldsymbol{\mu}_i = X\boldsymbol{\theta}_i, i = 1, \cdots, k$, where X is a $p \times q$ given matrix with rank q and $\boldsymbol{\theta}_i$'s are unknown parameter vectors. The model with such a mean structure is applied to growth curve data. Fujikoshi (2009) studied a likelihood ratio statistic for a parallelism hypothesis. In this paper we derive likelihood ratio statistics for level and flatness hypotheses under the parallelism hypothesis. Their null distributions are obtained. We also give an example.

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§1. Introduction

Consider k p-dimensional normal populations with Π_i : $N_p(\boldsymbol{\mu}_i, \Sigma), i = 1, \ldots, k$. Let $\boldsymbol{y}_{ij}, j = 1, \ldots, n_i$ be random samples from $N_p(\boldsymbol{\mu}_i, \Sigma)$. It is assumed that the observations follow the growth curve model (see, e.g., Potthoff and Roy [8], Kshirsagar and Smith [7]) given by

$$\boldsymbol{\mu}_i = X\boldsymbol{\theta}_i, \ \ i = 1, \ldots, k, \tag{1.1}$$

where X is a $p \times q$ given matrix with rank q, $\boldsymbol{\theta}_i$'s are unknown parameter vectors, and Σ is unknown.

One is interested in profile analysis of k p-dimensional populations. Without assuming the mean structures (1.1) some statistical methods have been proposed by Greenhouse and Geisser [6], Srivastava [11], [12], etc. Fujikoshi

[2] studied a likelihood ratio (LR) test for a parallelism hypothesis under the growth curve model (1.1). The parallelism hypothesis is expressed as

$$H_1 : X\boldsymbol{\theta}_i - X\boldsymbol{\theta}_k = \gamma_i \mathbf{1}_p, \quad i = 1, \ldots, k-1, \tag{1.2}$$

where γ_i's are unknown parameters and $\mathbf{1}_p$ is the p-dimensional vector with all the elements 1.

It is assumed that the first column of X is $\mathbf{1}_p$, i.e., $X = (\mathbf{1}_p, X_2)$. Then, we have $\mathbf{e}_q = (X'X)^{-1}X'\mathbf{1}_p = (1, 0, \cdots, 0)'$ since $(X'X)^{-1}X'\mathbf{1}_p$ is the first column of $(X'X)^{-1}X'X = I_q$. Using the property it is seen (see Fujikoshi [2]) that the parallelism hypothesis (1.2) is equivalent to the following hypothesis

$$\begin{aligned} H_1 : \quad & \boldsymbol{\theta}_i - \boldsymbol{\theta}_k = \gamma_i \boldsymbol{e}_q, \ i = 1, \ldots, k-1, \\ & \Leftrightarrow \boldsymbol{\theta}_{12} = \cdots = \boldsymbol{\theta}_{k2}, \end{aligned}$$

where $\boldsymbol{\theta}_j = (\theta_{j1}, \theta_{j2}, \cdots \theta_{jq})' = (\theta_{j1}, \boldsymbol{\theta}'_{j2})', \ j = 1, \ldots, k$.

In this paper we consider the level hypothesis under the parallelism hypothesis, which is defined by

$$H_2 \mid H_1 : \boldsymbol{\theta}_1 = \cdots = \boldsymbol{\theta}_k.$$

Further, we consider the flatness hypothesis under the parallelism hypothesis, which is

$$H_3 \mid H_1 : \boldsymbol{\theta}_{12} = \cdots = \boldsymbol{\theta}_{k2} = \mathbf{0}.$$

The level hypothesis under the parallelism hypothesis $H_2 \mid H_1$ means that the q profiles are coincident. The flatness hypothesis under the parallelism hypothesis $H_3 \mid H_1$ means that the q profiles are constants.

The purpose of this paper is to obtain LR tests for $H_2 \mid H_1$ and $H_3 \mid H_1$. Their null distributions are also obtained. In Section 2 we give expression of the two hypotheses in terms of a canonical form due to Fujikoshi [2]. In Section 3 we derive an LR test for $H_2 \mid H_1$ and its null distribution. In Section 3 we derive an LR test for $H_3 \mid H_1$ and its null distribution. An example is given in Section 4.

§2. Two hypotheses in canonical form

In this section we give a canonical form for testing two profile hypotheses $H_2 \mid H_1$ and $H_3 \mid H_1$, along Fujikoshi [2]. Let all the observations be denoted by

$$Y = (\boldsymbol{y}_{11}, \ldots, \boldsymbol{y}_{1n_1}, \boldsymbol{y}_{21}, \ldots, \boldsymbol{y}_{2n_2}, \ldots, \boldsymbol{y}_{k1}, \ldots, \boldsymbol{y}_{kn_k})'.$$

Then the growth curve model in (1.1) is expressed as

$$M : \ \mathrm{E}(Y) = A\Theta X', \tag{2.1}$$

where the rows of Y are independently distributed as p-variate normal distributions with the same covariance matrix Σ, $\Theta = (\boldsymbol{\theta}_1, \dots, \boldsymbol{\theta}_k)'$, and A is the design matrix between individuals given by

$$A = \begin{pmatrix} \mathbf{1}_{n_1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_{n_2} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{1}_{n_k} \end{pmatrix}.$$

The growth model (1.1) satisfying the parallelism hypothesis H_1 is expressed as

$$M_1 : \ \mathrm{E}(Y) = \mathbf{1}_n \boldsymbol{\theta}_k' X' + A_1 \boldsymbol{\gamma} \mathbf{1}_p',$$

where A_1 is a submatrix composed from the first $k-1$ columns of A.

We consider a transformation from Y to $Z = H'YB$, where H and B are the orthogonal matrices defined as follows: The orthogonal matrix $H = (\boldsymbol{h}_1, H_2, H_3)$ is defined by

$$\boldsymbol{h}_1 = (1/\sqrt{n})\mathbf{1}_n, \quad H_2 = (I_n - P_n)A_1\{A_1'(I_n - P_n)A_1\}^{-1/2},$$

and H_3 is an $n \times (n-k)$ matrix such that $H_3'A = O$ and $H_3'H_3 = I_{n-k}$. Here for a positive integer m, $P_m = (1/m)\mathbf{1}_m\mathbf{1}_m'$. The space generated by the column vectors of A and its orthogonal space are denoted by $\mathcal{R}[A]$ and $\mathcal{R}[A]^\perp$, respectively. The matrix H_3 satisfies $H_3H_3' = I_{n-k} - P_A$, where $P_A = A(A'A)^{-1}A'$. Similarly the orthogonal matrix $B = (\boldsymbol{b}_1, B_2, B_3)$ is defined by

$$\boldsymbol{b}_1 = (1/\sqrt{p})\mathbf{1}_p, \quad B_2 = (I_p - P_p)X_2\{X_2'(I_p - P_p)X_2\}^{-1/2},$$

and B_3 is a $p \times (p-q)$ matrix such that $B_3'X = O$ and $B_3'B_3 = I_{p-q}$. The transformed matrix Z is decomposed as

$$\begin{aligned} Z &= (\boldsymbol{h}_1, H_2, H_3)'Y(\boldsymbol{b}_1, B_2, B_3) \\ &= \begin{pmatrix} z_{11} & \boldsymbol{z}_{12}' & \boldsymbol{z}_{13}' \\ \boldsymbol{z}_{21} & Z_{22} & Z_{23} \\ \boldsymbol{z}_{31} & Z_{32} & Z_{33} \end{pmatrix}. \end{aligned}$$

Then, the mean of Z under (2.1) is

$$\begin{aligned} \mathrm{E}[Z] &= \mathrm{E}\left[\begin{pmatrix} z_{11} & \boldsymbol{z}_{12}' & \boldsymbol{z}_{13}' \\ \boldsymbol{z}_{21} & Z_{22} & Z_{23} \\ \boldsymbol{z}_{31} & Z_{32} & Z_{33} \end{pmatrix} \right] \\ &= \begin{pmatrix} \xi_{11} & \boldsymbol{\xi}_{12}' & \mathbf{0}' \\ \boldsymbol{\xi}_{21} & \Xi_{22} & O \\ \mathbf{0} & O & O \end{pmatrix} = \begin{pmatrix} \nu_1 & \boldsymbol{\nu}_2' & \mathbf{0}' \\ \boldsymbol{\delta} & \Xi_{22} & O \\ \mathbf{0} & O & O \end{pmatrix}, \end{aligned}$$

where

$$\begin{aligned}\boldsymbol{\nu} &= (\nu_1, \boldsymbol{\nu}_2')' \\ &= (\boldsymbol{b}_1, B_2)'\{\sqrt{n}X\boldsymbol{\theta}_k + n^{-1/2}(n_1\gamma_1 + \cdots + n_{k-1}\gamma_{k-1})\mathbf{1}_p\}, \\ \boldsymbol{\delta} &= \sqrt{p}\{A_1'(I_n - P_0)A_1\}^{1/2}\boldsymbol{\gamma}.\end{aligned}$$

The rows of Z are independently normal, and have the same covariance matrix given by

$$\begin{aligned}\Psi &= (\boldsymbol{b}_1, B_2, B_3)'\Sigma(\boldsymbol{b}_1, B_1, B_3) \\ &= \begin{pmatrix} \psi_{11} & \boldsymbol{\psi}_{21}' & \boldsymbol{\psi}_{31}' \\ \boldsymbol{\psi}_{21} & \Psi_{22} & \Psi_{23} \\ \boldsymbol{\psi}_{31} & \Psi_{32} & \Psi_{33} \end{pmatrix}.\end{aligned}$$

The parallelism hypothesis H_1 in the growth curve model (2.1) is expressed as

$$H_1 : \; \Xi_{22} = O. \tag{2.2}$$

Similarly the level hypothesis under H_1 is expressed as

$$H_2 \mid H_1 : \; \Xi_{22} = O, \; \boldsymbol{\delta} = \mathbf{0}, \tag{2.3}$$

and the flatness hypothesis under H_1 is expressed as

$$H_3 \mid H_1 : \; \Xi_{22} = O, \; \boldsymbol{\nu}_2 = \mathbf{0}.$$

§3. LR test for level hypothesis

The LR test for testing H_1 under the growth curve model (1.1) or (2.1) was obtained (Fujikoshi [2]) by using the canonical form in terms of Z. In this section we consider the likelihood function based on the distribution of Z. Let $L(\boldsymbol{\nu}, \boldsymbol{\delta}, \Xi_{22}, \Psi)$ be the likelihood under (1.1) or (2.1). Then, the likelihood under the parallelism hypothesis (2.2) is given by $L_1(\boldsymbol{\nu}, \boldsymbol{\delta}, \Psi) = L(\boldsymbol{\nu}, \boldsymbol{\delta}, O, \Psi)$. Therefore we have

$$\begin{aligned}g_1(\boldsymbol{\nu}, \boldsymbol{\delta}, \Psi) &\equiv -2\log L_1(\boldsymbol{\nu}, \boldsymbol{\delta}, \Psi) \\ &= n\log|\Psi| + np\log 2\pi \\ &\quad + \mathrm{tr}\Psi^{-1}\Big[(\boldsymbol{z}_{1(12)}' - \boldsymbol{\nu}', \boldsymbol{z}_{13}')'(\boldsymbol{z}_{1(12)}' - \boldsymbol{\nu}', \boldsymbol{z}_{13}') \\ &\quad + (\boldsymbol{z}_{21} - \boldsymbol{\delta}, Z_{2(23)})'(\boldsymbol{z}_{21} - \boldsymbol{\delta}, Z_{2(23)}) + W^{(1)}\Big].\end{aligned}$$

Here

$$\begin{aligned}
\boldsymbol{z}'_{1(12)} &= (z_{11}, \boldsymbol{z}'_{12}), \quad Z_{2(23)} = (Z_{22}, Z_{23}),\\
W^{(1)} &= (\boldsymbol{z}_{31}, Z_{32}, Z_{33})'(\boldsymbol{z}_{31}, Z_{32}, Z_{33})\\
&= \begin{pmatrix} w_{11}^{(1)} & (\boldsymbol{w}_{21}^{(1)})' & (\boldsymbol{w}_{31}^{(1)})' \\ \boldsymbol{w}_{21}^{(1)} & W_{22}^{(1)} & W_{23}^{(1)} \\ \boldsymbol{w}_{31}^{(1)} & W_{32}^{(1)} & W_{33}^{(1)} \end{pmatrix}.
\end{aligned}$$

Similar notations are used for partitions of Ψ.

Fujikoshi [2] showed that

$$\begin{aligned}
\min_{\boldsymbol{\nu},\boldsymbol{\delta},\Psi} g_1(\boldsymbol{\nu}, \boldsymbol{\delta}, \Psi) &= \min_{\psi_{11\cdot 23}, \Psi_{22\cdot 3}, \Psi_{33}} \Big[n \log\{\psi_{11\cdot 23} \cdot |\Psi_{22\cdot 3}| \cdot |\Psi_{33}|\} \\
&\quad + np \log 2\pi + \psi_{11\cdot 23}^{-1} w_{11\cdot 23}^{(1)} + \mathrm{tr}\Psi_{33}^{-1}(W_{33}^{(2)} + \boldsymbol{z}_{13}\boldsymbol{z}'_{13}) + \mathrm{tr}\Psi_{22\cdot 3}^{-1} W_{22\cdot 3}^{(2)} \Big] \qquad (3.1)\\
&= n \log\{\hat{\psi}_{11\cdot 23}^{(\Omega)} \cdot |\hat{\Psi}_{22\cdot 3}^{(\Omega)}| \cdot |\hat{\Psi}_{33}^{(\Omega)}|\} + np(\log 2\pi + 1),
\end{aligned}$$

where

$$\begin{aligned}
n\hat{\psi}_{11\cdot 23}^{(\Omega)} &= w_{11\cdot 23}^{(1)} = w_{11}^{(1)} - (\boldsymbol{w}_{(23)1}^{(1)})'(W_{(23)(23)}^{(1)})^{-1} \boldsymbol{w}_{(23)1}^{(1)},\\
n\hat{\Psi}_{22\cdot 3}^{(\Omega)} &= W_{22\cdot 3}^{(2)} = W_{22}^{(2)} - W_{23}^{(2)}(W_{33}^{(2)})^{-1} W_{32}^{(2)},\\
n\hat{\Psi}_{33}^{(\Omega)} &= W_{33}^{(2)} + \boldsymbol{z}_{13}\boldsymbol{z}'_{13},\\
W^{(2)} &= W^{(1)} + (\boldsymbol{z}_{21}, Z_{22}, Z_{23})'(\boldsymbol{z}_{21}, Z_{22}, Z_{23})\\
&= \begin{pmatrix} w_{11}^{(2)} & (\boldsymbol{w}_{21}^{(2)})' & (\boldsymbol{w}_{31}^{(2)})' \\ \boldsymbol{w}_{21}^{(2)} & W_{22}^{(2)} & W_{23}^{(2)} \\ \boldsymbol{w}_{31}^{(2)} & W_{32}^{(2)} & W_{33}^{(2)} \end{pmatrix}.
\end{aligned}$$

In order to derive an LR test for testing the level hypothesis $H_2 \mid H_1$, consider the likelihood $L_1(\boldsymbol{\nu}, \mathbf{0}, \Psi) \equiv L_2(\boldsymbol{\nu}, \Psi)$ under $H_2 \mid H_1$. We have

$$\begin{aligned}
g_2(\boldsymbol{\nu}, \Psi) \;\; \equiv \;\; & -2 \log L_2(\boldsymbol{\nu}, \Psi) = n \log |\Psi| + np \log 2\pi \\
& + \mathrm{tr}\Psi^{-1} \Big[(\boldsymbol{z}'_{1(12)} - \boldsymbol{\nu}', \boldsymbol{z}'_{13})'(\boldsymbol{z}'_{1(12)} - \boldsymbol{\nu}', \boldsymbol{z}'_{13}) \\
& + (\boldsymbol{z}_{21}, Z_{2(23)})'(\boldsymbol{z}_{21}, Z_{2(23)}) + W^{(1)} \Big].
\end{aligned}$$

We also use the following formula

$$\begin{aligned}
\Psi^{-1} &= \begin{pmatrix} \psi_{(12)(12)} & \psi_{(12)3} \\ \psi_{3(12)} & \psi_{33} \end{pmatrix}^{-1}\\
&= \begin{pmatrix} O & O \\ O & \psi_{33}^{-1} \end{pmatrix} + \begin{pmatrix} I_{p-q} \\ -\Psi_{33}^{-1}\Psi_{3(12)} \end{pmatrix} \Psi_{(12)(12)\cdot 3}^{-1} (I_{p-q}, \; -\Psi_{(12)3}\Psi_{33}^{-1}),
\end{aligned}$$

where $\Psi_{(12)(12)\cdot 3} = \Psi_{(12)(12)} - \Psi_{(12)3}\Psi_{33}^{-1}\Psi_{3(12)}$. Further, the following formulas are used in our derivation

$$
\begin{aligned}
&|\Psi| = \psi_{11\cdot 23} \cdot |\Psi_{(23)(23)}| = \psi_{11\cdot 23} \cdot |\Psi_{22\cdot 33}| \cdot |\Psi_{33}|, \\
&\mathrm{tr}\Psi^{-1}(\boldsymbol{z}'_{1(12)} - \boldsymbol{\nu}', \boldsymbol{z}'_{13})'(\boldsymbol{z}'_{1(12)} - \boldsymbol{\nu}', \boldsymbol{z}'_{13}) = \mathrm{tr}\Psi_{33}^{-1}\boldsymbol{z}_{13}\boldsymbol{z}'_{13} \\
&\qquad + \mathrm{tr}\Psi_{(12)(12)\cdot 3}^{-1}(\boldsymbol{z}'_{1(12)} - \boldsymbol{\nu}' - \boldsymbol{z}'_{13}\mathcal{C})'(\boldsymbol{z}'_{1(12)} - \boldsymbol{\nu}' - \boldsymbol{z}'_{13}\mathcal{C}), \\
&\mathrm{tr}\Psi^{-1}(\boldsymbol{z}_{21}, Z_{2(23)})'(\boldsymbol{z}_{21}, Z_{2(23)}) = \mathrm{tr}\Psi_{(23)(23)}^{-1} Z'_{2(23)} Z_{2(23)} \\
&\qquad + \psi_{11\cdot 23}^{-1}(\boldsymbol{z}_{21} - Z_{2(23)}\boldsymbol{\eta})'(\boldsymbol{z}_{21} - Z_{2(23)}\boldsymbol{\eta}), \\
&\mathrm{tr}\Psi^{-1}W^{(1)} = \mathrm{tr}\Psi_{(23)(23)}^{-1} W_{(23)(23)}^{(1)} \\
&\qquad + \psi_{11\cdot 23}^{-1}(\boldsymbol{z}_{31} - Z_{3(23)}\boldsymbol{\eta})'(\boldsymbol{z}_{31} - Z_{3(23)}\boldsymbol{\eta}),
\end{aligned}
$$

where $\mathcal{C} = \Psi_{33}^{-1}\Psi_{3(12)}$ and $\boldsymbol{\eta} = \Psi_{(23)(23)}^{-1}\boldsymbol{\psi}_{(23)1}$.

Note that there is a one-to-one correspondence between Ψ and $\{\Psi_{(23)(23)}, \psi_{11\cdot 23}, \boldsymbol{\eta}\}$. Similarly there is a one-to-one correspondence between $\Psi_{(23)(23)}$ and $\{\Psi_{33}, \Psi_{22\cdot 3}, \mathcal{B}\}$, where $\mathcal{B} = \Psi_{33}^{-1}\Psi_{32}$. Furthermore, $\mathcal{C}$ is a function of $\boldsymbol{\eta}$ and $\mathcal{B}$. In fact, $\mathcal{C} = (\Psi_{33}^{-1}\boldsymbol{\psi}_{31}, \mathcal{B})$. From the definition of $\boldsymbol{\eta}$ we have $\Psi_{(23)(23)}\boldsymbol{\eta} = \boldsymbol{\psi}_{(23)1}$. Multiplying from the left by $\Psi_{33}^{-1}(O, I_{p-q})$ to both sides of the equality, we have $(\mathcal{B}, I_{p-q})\boldsymbol{\eta} = \Psi_{33}^{-1}\boldsymbol{\psi}_{31}$ which leads to

$$\mathcal{C} = ((\mathcal{B}, I_{p-q})\boldsymbol{\eta}, \mathcal{B})\,.$$

Therefore the minimization with respect to $\mathcal{C}$ should be considered through the minimization with respect to $\boldsymbol{\eta}$ and $\mathcal{B}$, and $\hat{\mathcal{C}}$ is determined from $\hat{\boldsymbol{\eta}}$ and $\hat{\mathcal{B}}$.

It is easy to see that the minimization of $g_2(\boldsymbol{\mu}, \Psi)$ with respect to $\boldsymbol{\mu}$ and $\boldsymbol{\eta}$ is attained at

$$
\begin{aligned}
\hat{\boldsymbol{\nu}} &= \boldsymbol{z}_{1(12)} - \hat{\mathcal{C}}'\boldsymbol{z}_{13}, \\
\hat{\boldsymbol{\eta}} &= (Z'_{2(23)}Z_{2(23)} + Z'_{3(23)}Z_{3(23)})^{-1}(Z'_{2(23)}\boldsymbol{z}_{21} + Z'_{3(23)}\boldsymbol{z}_{31}) \\
&= \{W_{(23)(23)}^{(2)}\}^{-1}\boldsymbol{w}_{(23)1}^{(2)}.
\end{aligned}
$$

Note that

$$
\begin{aligned}
&\min_{\boldsymbol{\eta}}(\boldsymbol{z}_{21} - Z_{2(23)}\boldsymbol{\eta})'(\boldsymbol{z}_{21} - Z_{2(23)}\boldsymbol{\eta}) + (\boldsymbol{z}_{31} - Z_{3(23)}\boldsymbol{\eta})'(\boldsymbol{z}_{31} - Z_{3(23)}\boldsymbol{\eta}) \\
&\qquad = w_{11\cdot 23}^{(2)}.
\end{aligned}
$$

These imply that

$$
\begin{aligned}
\min_{\boldsymbol{\nu},\Psi} g_2(\boldsymbol{\nu}, \Psi) = &\min_{\psi_{11\cdot 23}, \Psi_{(23)(23)}} \left[n \log\{\psi_{11\cdot 23} \cdot |\Psi_{(23)(23)}|\}\right. \\
&+ \psi_{11\cdot 23}^{-1} w_{11\cdot 23}^{(2)} + np\log 2\pi + \mathrm{tr}\Psi_{33}^{-1}\boldsymbol{z}_{13}\boldsymbol{z}'_{13} \\
&\left.+ \mathrm{tr}\Psi_{(23)(23)}^{-1}\left\{W_{(23)(23)}^{(1)} + Z'_{2(23)}Z_{2(23)}\right\}\right].
\end{aligned}
\tag{3.2}
$$

Note that

$$\begin{aligned}
&\mathrm{tr}\Psi^{-1}_{(23)(23)}W^{(2)}_{(23)(23)} = \Psi^{-1}_{33}W^{(2)}_{33} \\
&\qquad +\mathrm{tr}\Psi^{-1}_{22\cdot 3}\left\{(W^{(2)-1}_{33}W^{(2)}_{32} - \mathcal{B})'W^{(2)}_{33}(W^{(2)-1}_{33}W^{(2)}_{32} - \mathcal{B}) + W^{(2)}_{22\cdot 3}\right\}.
\end{aligned} \tag{3.3}$$

Substituting (3.3) to (3.2),

$$\begin{aligned}
&\min_{\boldsymbol{\nu},\Psi} g_2(\boldsymbol{\nu},\Psi) = \min_{\psi_{11\cdot 23},\Psi_{22\cdot 3},\Psi_{33}} [n\log\{\psi_{11\cdot 23}\cdot|\Psi_{22\cdot 3}|\cdot|\Psi_{33}|\} \\
&\qquad +np\log 2\pi + \psi^{-1}_{11\cdot 23}w^{(2)}_{11\cdot 23} + \mathrm{tr}\Psi^{-1}_{33}(W^{(2)}_{33} + \boldsymbol{z}_{13}\boldsymbol{z}'_{13}) + \mathrm{tr}\Psi^{-1}_{22\cdot 3}W^{(2)}_{22\cdot 3}\Big] \\
&\quad = n\log\{\hat{\psi}^{(\omega)}_{11\cdot 23}\cdot|\hat{\Psi}^{(\omega)}_{22\cdot 3}|\cdot|\hat{\Psi}^{(\omega)}_{33}|\} + np(\log 2\pi + 1),
\end{aligned} \tag{3.4}$$

where

$$\begin{aligned}
n\hat{\psi}^{(\omega)}_{11\cdot 23} &= w^{(2)}_{11\cdot 23},\ n\hat{\Psi}^{(\omega)}_{22\cdot 3} = W^{(2)}_{22\cdot 3}, \\
n\hat{\Psi}^{(\omega)}_{33} &= W^{(2)}_{33} + \boldsymbol{z}_{13}\boldsymbol{z}'_{13} = W^{(3)}_{33}.
\end{aligned}$$

From (3.4) and (3.1) we have a likelihood ratio statistic given by

$$\lambda_2^{2/n} = L_{(2)} = \frac{\hat{\psi}^{(\Omega)}_{11\cdot 23}\cdot|\hat{\Psi}^{(\Omega)}_{22\cdot 3}|\cdot|\hat{\Psi}^{(\Omega)}_{33}|}{\hat{\psi}^{(\omega)}_{11\cdot 23}\cdot|\hat{\Psi}^{(\omega)}_{22\cdot 3}|\cdot|\hat{\Psi}^{(\omega)}_{33}|} = \frac{w^{(1)}_{11\cdot 23}}{w^{(2)}_{11\cdot 23}}. \tag{3.5}$$

We show that the null distribution of $\lambda_2^{2/n}$ is expressed as a lambda distribution, which is defined as follows: Suppose that $B \sim W_p(q,\Sigma)$, $W \sim W_p(n,\Sigma)$, and B and W are independent. Then the distribution of $\Lambda = |W|/|B+W|$ is called a lambda distribution with parameter (p,q,n), which is denoted as $\Lambda \sim \Lambda_p(q,n)$. When p or q is 1 or 2, the exact distribution of Λ is known. For the result, see, e.g., Anderson [1]. In general, we have a chi-square approximation given by

$$-[n + \frac{1}{2}(q-p-1)]\log\Lambda \ \approx\ \chi^2_{pq}.$$

In order to derive the null distribution of $\lambda_2^{2/n}$, we use the following well known Lemma (see, e.g., Fujikoshi et al. [4]).

Lemma 3.1. *Suppose that $B \sim W_p(q,\Sigma)$, $W \sim W_p(n,\Sigma)$, and B and W are independent. Let B, W and $T = B + W$ be decomposed as $k, p-k$ rows and columns, i.e.*

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22}\end{pmatrix},\quad W = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22}\end{pmatrix},\quad T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22}\end{pmatrix}.$$

Then

$$\Lambda_{2\cdot 1} = \frac{|W_{22\cdot 1}|}{|T_{22\cdot 1}|} \sim \Lambda_{p-k}(q, n-k).$$

Theorem 3.1. *The LR criterion λ_2 for the level hypothesis $H_2 \mid H_1$ in* (2.3) *under the growth curve model is given by* (3.5). *Further, the null distribution of $L_{(2)}$ is a lambda distribution $\Lambda_1(k-1, n-k-p+1)$.*

Proof. In the first half of this section we have proved that λ_2 is given by (3.5), so we prove the distributional result. Note that

$$W^{(1)} = (\boldsymbol{z}_{31}, Z_{32}, Z_{33})'(\boldsymbol{z}_{31}, Z_{32}, Z_{33})$$

and

$$W^{(2)} = W^{(1)} + (\boldsymbol{z}_{21}, Z_{22}, Z_{23})'(\boldsymbol{z}_{21}, Z_{22}, Z_{23})$$

are distributed as Wishart distributions $W_p(n-k, \Sigma)$ and $W_p(n-k+k-1, \Sigma)$. Further, $W^{(1)}$ and $W^{(2)} - W^{(1)}$ are independent. Then, using Lemma 3.1, we obtain that $L_2 \sim \Lambda_1(k-1, n-k-(q-1+p-q))$. □

Now we consider to express $L_{(2)}$ in terms of the original observation matrix Y. Let S_w and S_b be the variation matrices due to within groups and between groups, respectively. They are given as

$$S_w = \sum_{i=1}^{k}\sum_{j=1}^{n_i}(\boldsymbol{y}_{ij} - \bar{\boldsymbol{y}}_i)(\boldsymbol{y}_{ij} - \bar{\boldsymbol{y}}_i)', \quad S_b = \sum_{i=1}^{k} n_i(\bar{\boldsymbol{y}}_i - \bar{\boldsymbol{y}})(\bar{\boldsymbol{y}}_i - \bar{\boldsymbol{y}})',$$

where $\bar{\boldsymbol{y}}_i = (1/n_i)\sum_{j=1}^{n_i} \boldsymbol{y}_{ij}$, $\bar{\boldsymbol{y}} = (1/n)\sum_{i=1}^{k}\sum_{j=1}^{n_i} \boldsymbol{y}_{ij}$, and $n = n_1 + \cdots + n_k$. Put

$$S_t = S_b + S_w = \sum_{i=1}^{k}\sum_{j=1}^{n_i}(\boldsymbol{y}_{ij} - \bar{\boldsymbol{y}})(\boldsymbol{y}_{ij} - \bar{\boldsymbol{y}})'.$$

Then we can show that

$$W^{(1)} = (\boldsymbol{b}_1, B_2, B_3)'S_w(\boldsymbol{b}_1, B_2, B_3), \quad W^{(2)} = (\boldsymbol{b}_1, B_2, B_3)'S_t(\boldsymbol{b}_1, B_2, B_3).$$

Therefore

$$\begin{aligned} w^{(1)}_{11\cdot 23} &= w^{(1)}_{11} - (\boldsymbol{w}^{(1)}_{(23)1})'(W^{(1)}_{(23)(23)})^{-1}\boldsymbol{w}^{(1)}_{(23)1} \\ &= \boldsymbol{b}_1'S_w\boldsymbol{b}_1 - \boldsymbol{b}_1'S_w\left[(B_2, B_3)\left\{(B_2, B_3)'S_w(B_2, B_3)\right\}^{-1}(B_2, B_3)'\right]S_w\boldsymbol{b}_1 \\ &= \boldsymbol{b}_1'S_w\boldsymbol{b}_1 - \boldsymbol{b}_1'S_w\left\{S_w^{-1} - S_w^{-1}\boldsymbol{b}_1(\boldsymbol{b}_1'S_w^{-1}\boldsymbol{b}_1)^{-1}\boldsymbol{b}_1'S_w^{-1}\right\}S_w\boldsymbol{b}_1 \\ &= \boldsymbol{b}_1'\boldsymbol{b}_1 \cdot (\boldsymbol{b}_1'S_w^{-1}\boldsymbol{b}_1)^{-1} \cdot \boldsymbol{b}_1'\boldsymbol{b}_1 = \frac{p}{\mathbf{1}_p'S_w^{-1}\mathbf{1}_p}. \end{aligned}$$

Similarly we have $w^{(2)}_{11\cdot 23} = p/(\mathbf{1}_p'S_w^{-1}\mathbf{1}_p)$. Therefore,

$$L_{(2)} = \frac{\mathbf{1}_p'S_t^{-1}\mathbf{1}_p}{\mathbf{1}_p'S_w^{-1}\mathbf{1}_p}.$$

This shows that $L_{(2)}$ is the same as an LR test (Srivastava [11], [12]) for coincidence test under parallel hypothesis in MANOVA model.

§4. LR test for flatness hypothesis

Let the likelihood of Z under $H_3 \mid H_1$ be denoted by $L_3(\nu_1, \boldsymbol{\delta}, \Psi)$ which is given by $L_3(\boldsymbol{\nu}, \boldsymbol{\delta}, \Psi)$ with $\boldsymbol{\nu}_2 = \mathbf{0}$. Then

$$\begin{aligned} g_3(\nu_1, \boldsymbol{\delta}, \Psi) \quad \equiv \quad & -2\log L_3(\nu_1, \boldsymbol{\delta}, \Psi) = n\log|\Psi| + np\log 2\pi \\ & +\mathrm{tr}\Psi^{-1}\Big[(z_{11} - \nu_1, \boldsymbol{z}'_{1(23)})'(z_{11} - \nu_1, \boldsymbol{z}'_{1(23)}) \\ & +(\boldsymbol{z}_{21} - \boldsymbol{\delta}, Z_{2(23)})'(\boldsymbol{z}_{21} - \boldsymbol{\delta}, Z_{2(23)}) + W^{(1)}\Big], \end{aligned}$$

where $\boldsymbol{z}'_{1(12)} = (z_{11}, \boldsymbol{z}'_{12})$ and $Z_{2(23)} = (Z_{22}, Z_{23})$. We also use the following notation and formula

$$\begin{aligned} \Psi^{-1} \quad = \quad & \begin{pmatrix} \psi_{11} & \boldsymbol{\psi}'_{(23)1} \\ \boldsymbol{\psi}_{(23)1} & \Psi_{(23)(23)} \end{pmatrix}^{-1} = \begin{pmatrix} O & O \\ O & \Psi^{-1}_{(23)(23)} \end{pmatrix} \\ & + \begin{pmatrix} 1 \\ -\Psi^{-1}_{(23)(23)}\boldsymbol{\psi}_{(23)1} \end{pmatrix} \psi^{-1}_{11\cdot 23}(1,\ -\boldsymbol{\psi}'_{(23)1}\Psi^{-1}_{(23)(23)}). \end{aligned}$$

In addition to the first and the last formulas in (3.2) the following formulas are used in our derivation.

$$\begin{aligned} & \mathrm{tr}\Psi^{-1}(z_{11} - \nu_1, \boldsymbol{z}'_{1(23)})'(z_{11} - \nu_1, \boldsymbol{z}'_{1(23)}) = \mathrm{tr}\Psi^{-1}_{(23)(23)}\boldsymbol{z}_{1(23)}\boldsymbol{z}'_{1(23)} \\ & \qquad +\mathrm{tr}\psi^{-1}_{11\cdot 23}(z_{11} - \nu_1 - Z'_{1(23)}\boldsymbol{\eta})'(z_{11} - \nu_1 - Z'_{1(23)}\boldsymbol{\eta}), \\ & \mathrm{tr}\Psi^{-1}(\boldsymbol{z}_{21} - \boldsymbol{\delta}, Z_{2(23)})'(\boldsymbol{z}_{21} - \boldsymbol{\delta}, Z_{2(23)}) = \mathrm{tr}\Psi^{-1}_{(23)(23)}Z'_{2(23)}Z_{2(23)} \\ & \qquad +\psi^{-1}_{11\cdot 23}(\boldsymbol{z}_{21} - \boldsymbol{\delta} - Z_{2(23)}\boldsymbol{\eta})'(\boldsymbol{z}_{21} - \boldsymbol{\delta} - Z_{2(23)}\boldsymbol{\eta}). \end{aligned}$$

By an analogy to the method of minimizing $g_1(\boldsymbol{\mu}, \boldsymbol{\delta}, \Psi)$ it is easy to see that the minimization of $g_3(\mu_1, \boldsymbol{\delta}, \Psi)$ with respect to $\mu_1, \boldsymbol{\delta}$ and $\boldsymbol{\eta}$ is attained at

$$\begin{aligned} & \hat{\boldsymbol{\delta}} = \boldsymbol{z}_{21} - Z_{2(23)}\hat{\boldsymbol{\eta}}, \quad \hat{\nu_1} = z_{11} - \boldsymbol{z}'_{1(23)}\hat{\boldsymbol{\eta}}, \\ & \hat{\boldsymbol{\eta}} = (Z'_{3(23)}Z_{3(23)})^{-1}Z'_{3(23)}\boldsymbol{z}_{31} = \{W^{(1)}_{(23)(23)}\}^{-1}\boldsymbol{w}^{(1)}_{(23)1}. \end{aligned}$$

These imply that

$$\begin{aligned} \min_{\nu_1, \boldsymbol{\delta}, \Psi} g_3(\boldsymbol{\nu}, \boldsymbol{\delta}, \Psi) = & \min_{\psi_{11\cdot 23}, \Psi_{(23)(23)}} \big[n\log\{\psi_{11\cdot 23}\cdot|\Psi_{(23)(23)}|\} \\ & +\psi^{-1}_{11\cdot 23}w^{(1)}_{11\cdot 23} + np\log 2\pi + \mathrm{tr}\Psi^{-1}_{(23)(23)}\boldsymbol{z}_{1(23)}\boldsymbol{z}'_{1(23)} \\ & +\mathrm{tr}\Psi^{-1}_{(23)(23)}Z'_{2(23)}Z_{2(23)} + \mathrm{tr}\Psi^{-1}_{(23)(23)}W^{(1)}_{(23)(23)}\Big]. \end{aligned}$$

Here we use

$$\begin{aligned} & \min_{\eta}(\boldsymbol{z}_{31} - Z_{3(23)}\boldsymbol{\eta})'(\boldsymbol{z}_{31} - Z_{3(23)}\boldsymbol{\eta}) = (\boldsymbol{w}^{(1)}_{31})'(I_{n-k} - P_{Z_{3(23)}})\boldsymbol{w}^{(1)}_{31} \\ & \qquad = w^{(1)}_{11\cdot 3} - (\boldsymbol{w}^{(1)}_{21\cdot 3})'W^{-1}_{22\cdot 3}\boldsymbol{w}^{(1)}_{21\cdot 3} = w^{(1)}_{11\cdot 23}. \end{aligned}$$

Let

$$\begin{aligned} W^{(3)} &= W^{(2)} + (\boldsymbol{z}_{21}, Z_{22}, Z_{23})'(\boldsymbol{z}_{21}, Z_{22}, Z_{23}) \\ &= \begin{pmatrix} w_{11}^{(3)} & (\boldsymbol{w}_{21}^{(3)})' & (\boldsymbol{w}_{31}^{(3)})' \\ \boldsymbol{w}_{21}^{(3)} & W_{22}^{(3)} & W_{23}^{(3)} \\ \boldsymbol{w}_{31}^{(3)} & W_{32}^{(3)} & W_{33}^{(3)} \end{pmatrix}. \end{aligned}$$

Then we have

$$\begin{aligned} \min_{\nu_1,\boldsymbol{\delta},\Psi} g_3(\nu_1,\boldsymbol{\delta},\Psi) &= \min_{\psi_{11\cdot 23},\Psi_{22\cdot 3},\Psi_{33}} \Big[n\log\{\psi_{11\cdot 23}\cdot|\Psi_{22\cdot 3}|\cdot|\Psi_{33}|\} \\ &\quad + np\log 2\pi + \psi_{11\cdot 23}^{-1} w_{11\cdot 23}^{(1)} + \mathrm{tr}\Psi_{22\cdot 3}^{-1} W_{22\cdot 3}^{(3)} + \mathrm{tr}\Psi_{33}^{-1} W_{33}^{(3)} \Big] \\ &= n\log\{\hat{\psi}_{11\cdot 23}^{(\tau)}\cdot|\hat{\Psi}_{22\cdot 3}^{(\tau)}|\cdot|\hat{\Psi}_{33}^{(\tau)}|\} + np(\log 2\pi + 1), \end{aligned} \tag{4.1}$$

where

$$n\hat{\psi}_{11\cdot 23}^{(\tau)} = w_{11\cdot 23}^{(1)},\ n\hat{\Psi}_{22\cdot 3}^{(\tau)} = W_{22\cdot 3}^{(3)},\ n\hat{\Psi}_{33}^{(\tau)} = W_{33}^{(3)}.$$

Using (4.1) and (3.1), we obtain a likelihood ratio statistic given in the following theorem.

Theorem 4.1. *The LR criterion λ_3 for H_3 in* (1.2) *under the growth curve model* (1.1) *is given by*

$$\begin{aligned} \lambda_3^{2/n} = L_{(3)} &= \frac{\hat{\psi}_{11\cdot 23}^{(\Omega)}\cdot|\hat{\Psi}_{22\cdot 3}^{(\Omega)}|\cdot|\hat{\Psi}_{33}^{(\Omega)}|}{\hat{\psi}_{11\cdot 23}^{(\tau)}\cdot|\hat{\Psi}_{22\cdot 3}^{(\tau)}|\cdot|\hat{\Psi}_{33}^{(\tau)}|} \\ &= \frac{|W_{22\cdot 3}^{(2)}|}{|W_{22\cdot 3}^{(3)}|}. \end{aligned}$$

Further, the null distribution of $L_{(3)}$ is $\Lambda_{q-1}(1, n-1-(p-q))$.

Proof. The likelihood test statistic is obtained from (4.1) and (3.1). In order to obtain the null distribution of $\lambda_3^{2/n}$, we note that $W_{(23)(23)}^{(2)} = W_{(23)(23)}^{(1)} + (Z_{22}, Z_{23})'(Z_{22}, Z_{23})$ and

$$W_{(23)(23)}^{(3)} = W_{(23)(23)}^{(2)} + (\boldsymbol{z}_{12}', \boldsymbol{z}_{13}')'(\boldsymbol{z}_{12}', \boldsymbol{z}_{13}')$$

are distributed as Wishart distributions $W_{q-1+p-q}(n-k+k-1, \Sigma_{(23)(23)})$ and $W_{p-1}(n-1+1, \Sigma_{(23)(23)})$, respectively. Further, $W^{(2)}$ and $W_{(23)(23)}^{(3)} - W_{(23)(23)}^{(2)}$ are independent. Then, using Lemma 3.1 we have

$$L_{(3)} \sim \Lambda_{q-1}(1, n-1-(p-q)).$$

□

Since $L_{(3)} \sim \Lambda_{q-1}(1, n-1-(p-q))$, it holds (see, e.g., Anderson [1]) that under $H_3 \mid H_1$

$$\frac{1-L_{(3)}}{L_{(3)}} \cdot \frac{n-p+1}{q-1} \sim F_{q-1,n-p+1}.$$

The statistic $L_{(3)}$ may be computed as follows. Note that

$$W^{(3)} = (\boldsymbol{b}_1, B_2, B_3)'(S_t + n\bar{\boldsymbol{y}}\bar{\boldsymbol{y}}')(\boldsymbol{b}_1, B_2, B_3).$$

Therefore, we have

$$\begin{aligned} W^{(2)}_{22\cdot 3} &= B_2'S_tB_2 - B_2'S_tB_3(B_3'S_tB_3)^{-1}B_3'S_tB_2, \\ W^{(3)}_{22\cdot 3} &= B_2'(S_t + n\bar{\boldsymbol{y}}\bar{\boldsymbol{y}}')B_2 - B_2'(S_t + n\bar{\boldsymbol{y}}\bar{\boldsymbol{y}}')B_3 \\ &\quad \times\{B_3'(S_t + n\bar{\boldsymbol{y}}\bar{\boldsymbol{y}}')B_3\}^{-1}B_3'(S_t + n\bar{\boldsymbol{y}}\bar{\boldsymbol{y}}')B_2. \end{aligned}$$

§5. Example

Before testing the level and the flatness hypotheses under a growth curve model, it needs to check whether (i) a growth curve model is appropriate or not, and (ii) the parallel hypothesis under a growth curve model is appropriate. In general, consider k p-dimensional normal populations with $\Pi_i :\ N_p(\boldsymbol{\mu}_i, \Sigma), i = 1, \ldots, k$. Let $\boldsymbol{y}_{ij}, j = 1, \ldots, n_{ij}$ be random samples from $N_p(\boldsymbol{\mu}_i, \Sigma)$. Consider to test

$$H_0 :\ \boldsymbol{\mu}_i = X\boldsymbol{\theta}_i,\ i = 1, \ldots, k,$$

which is a growth curve model given in (1.1). The likelihood ratio test is based on

$$\Lambda_0 = \frac{|\tilde{X}'S_w\tilde{X}|}{|\tilde{X}'S_w\tilde{X} + \tilde{X}'\sum_{i=1}^k n_i\bar{\boldsymbol{y}}_i\bar{\boldsymbol{y}}_i'\tilde{X}|} \tag{5.1}$$

whose null distribution is $\Lambda_{p-q}(k, n-k)$, where $\tilde{X}$ is a $p \times (p-q)$ matrix satisfying

$$I_p - X(X'X)^{-1}X = \tilde{X}\tilde{X}' \text{ and } \tilde{X}'\tilde{X} = I_{p-q}.$$

Next we consider to test the parallelism hypothesis

$$H_1 : X\boldsymbol{\theta}_i - X\boldsymbol{\theta}_k = \gamma_i\boldsymbol{1}_p, \quad i = 1, \ldots, k-1,$$

which is given in (1.2). The parallelism hypothesis is expressed as

$$C\Theta D = O,$$

where

$$C = (I_{k-1}, -\boldsymbol{1}_{k-1}), \quad D = \begin{pmatrix} \boldsymbol{0}' \\ I_{q-1} \end{pmatrix}.$$

It is known (see, e.g., Kshirsagar and Smith [7], Fujikoshi [2], [3], etc.) that the likelihood ratio test is based on

$$\Lambda_1 = \frac{|S_e|}{|S_e + S_h|}, \tag{5.2}$$

where

$$S_e = D'(X'S_w^{-1}X)^{-1}D, \quad S_h = (C\hat{\Theta}D)'(CRC')^{-1}C\hat{\Theta}D,$$

and $n = n_1 + \cdots + n_k$, $\bar{\boldsymbol{y}}_i = (1/n_i)\sum_{j=1}^{n_i} \boldsymbol{y}_{ij}$, i=1, …, k,

$$\begin{aligned}\hat{\Theta} &= (\bar{\boldsymbol{y}}_1, \ldots, \bar{\boldsymbol{y}}_k)'S_w^{-1}X(X'S_wX)^{-1}, \\ R &= \mathrm{diag}(\frac{1}{n_1}, \ldots, \frac{1}{n_k}) + (\bar{\boldsymbol{y}}_1, \ldots, \bar{\boldsymbol{y}}_k)'S_w^{-1} \\ &\quad \times \{S_w - X(X'S_w^{-1}X)^{-1}X'\}S_w^{-1}(\bar{\boldsymbol{y}}_1, \ldots, \bar{\boldsymbol{y}}_k).\end{aligned}$$

The null distribution of Λ_1 is a lambda distribution $\Lambda_{q-1}(k-1, n-k-(p-q))$.

Goldstein [5] applied a growth curve model to the data of height measurements for girls 6 $\sim$ 10 ages, which were classified into three groups (high, middle, low) according to the heights of their parents. Now we consider the measurements for two groups (middle, low), which are given in Table 5.1. The

Table 5.1. Height measurements for girls (M: Middle, L: Low)

Group	6 age	7 age	8 age	9 age	10 age
M	116.0	122.0	126.6	132.6	137.7
M	117.6	123.2	129.3	134.5	138.9
M	121.0	127.3	134.5	139.9	145.4
M	114.5	119.0	124.0	130.0	135.1
M	117.4	123.2	129.5	134.5	140.0
M	113.7	119.7	125.3	130.1	135.9
M	113.6	119.1	124.8	130.8	136.3
L	111.0	116.4	121.7	126.3	130.5
L	110.0	115.8	121.5	126.6	131.4
L	113.7	119.7	125.3	130.1	136.0
L	114.0	118.9	124.6	129.1	134.0
L	114.5	122.0	126.4	131.2	135.0
L	112.0	117.3	124.4	129.2	135.2

sample sizes are $n_1 = 7$ and $n_2 = 6$. Assume a growth curve model with a second degree polynomial, i.e., $\boldsymbol{\mu}_i = X\boldsymbol{\theta}_i, i = 1, 2$, where

$$X = \begin{pmatrix} 1 & 6 & 6^2 \\ 1 & 7 & 7^2 \\ 1 & 8 & 8^2 \\ 1 & 9 & 9^2 \\ 1 & 10 & 10^2 \end{pmatrix}.$$

In the growth curve model, $p = 5, q = 3, k = 2, n = 13$. Using (5.1), we have, as a test for growth curve model, $\Lambda_0 = 0.744806$, and

$$\frac{1 - \sqrt{\Lambda_0}}{\sqrt{\Lambda_0}} \cdot \frac{20}{4} = 0.7936 < F_{4,20}(0.05) = 2.8660.$$

Thus we do not reject the growth curve model. To test the parallelism hypothesis under the growth curve model, we use an LR test statistic (5.2). Then, $\Lambda_1 = 0.5634$, and

$$\frac{1 - \sqrt{\Lambda_1}}{\sqrt{\Lambda_1}} \cdot \frac{16}{2} = 2.6583 < F_{2,16}(0.05) = 3.6337.$$

Thus we do not reject the parallelism hypothesis. For testing the level hypothesis under the parallelism hypothesis, we have $\Lambda_2 \equiv L_{(2)} = 0.1480$, and

$$\frac{1 - \Lambda_2}{\Lambda_2} \cdot \frac{17}{1} = 97.8824 > F_{1,17}(0.01) = 8.3997.$$

Thus the level hypothesis is rejected. For testing the flatness hypothesis under the parallelism hypothesis, we have $\Lambda_3 \equiv L_{(3)} = 0.0081$, and

$$\frac{1 - \Lambda_3}{\Lambda_3} \cdot \frac{9}{2} = 549.745 > F_{2,9}(0.01) = 8.0215.$$

Thus the flatness hypothesis is also rejected.

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